

# Dimesoatoms production in high energy collisions

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## Abstract

The production of two meson electromagnetic bound states and free meson pairs  $\pi^+\pi^-$ ,  $K^+K^-$ ,  $\pi^+K^\mp$  in relativistic collisions has been considered. It was shown that making use of the exact Coulomb wave function for dimesoatom (DMA) allows one to calculate the yield of any  $nS$  state with desired accuracy. The relative probabilities of production of DMA and meson pairs in the free state are estimated. The amplitude of DMA transition from  $1S$  to  $2P$  state, which is essential for the ponium Lamb shift measurements, has been obtained.

## 1 Introduction

More than fifty years ago it was shown that the low energy scattering properties of strongly interacting charged particles are related to the hadronic properties of hydrogenlike atoms formed by such particles [1]. In particular the investigation of the ground state lifetime  $\tau_0$  of hadronic atoms and their Lamb shifts allow one to determine scattering lengths of meson-meson scattering [2]. There were considered all details of experiment on observation and study of such atoms produced in inclusive high-energy interactions. An approach for DMA ground state lifetime determination based on description of DMA as a multilevel system propagating and interacting in a target was developed in [4]. For the first time, experiment on the  $\pi^+\pi^-$  atoms study was done at Protvino at U-70 accelerator [5] and then continue at Proton Synchrotron at CERN [6], and the lifetime of DMA ground state  $\tau_0$  has been obtained [7]. This lifetime is dominated by the annihilation process  $\pi^+\pi^- \rightarrow \pi^0\pi^0$  and predicted most accurately in the Chiral Perturbation Theory [8–11]. Recently the new data on production of  $\pi^+K^\mp$  atoms [12] and first observation of long-lived ponium atoms have been reported [13].

All this stimulated us to consider in detail the production of bound and unbound states for any mesons. We compare the probability of meson pairs production in discrete and continuous states and estimate the accuracy of “zero radius” approximation which widely exploited in such type considerations. The amplitude for DMA transition from 1S to 2P state is derived in an analytic form. This transition is the main source of 2P states that is crucial for the Lamb shift measurements [14, 15] in experiments with dimesoatoms.

## 2 Dimesoatoms creation in inclusive processes

Let us consider the process of production of a pair of two oppositely charged mesons  $h^+, h^-$  in the inclusive process

$$a + b \rightarrow h^+ + h^- + X, \quad (1)$$

where  $X$  are particles whose momenta distribution is not essential in the later on consideration. In the further consideration The matrix element of the process (1)

$$M(\vec{p}_+, \vec{p}_-; \{\vec{p}_x\}) \quad (2)$$

is a function of  $h^+, h^-$ , momenta  $p_+, p_-$ , and the momenta of accompanying particles  $p_x$ . The invariant distribution of hadrons  $h^+, h^-$  has the usual form

$$(2\pi)^6 2E_+ \cdot 2E_- \frac{d\sigma}{d\vec{p}_+ d\vec{p}_-} = \frac{1}{4\sqrt{(p_a p_b)^2 - m_a^2 m_b^2}} \times \int |M(\vec{p}_+, \vec{p}_-; \{\vec{p}_x\})|^2 (2\pi)^4 \delta^4(p_a + p_b - p_+ - p_- - p_x) d\Phi_x, \quad (3)$$

where the phase space volume reads

$$d\Phi_x = \prod_{i=1}^{N(X)} \frac{d^3 \vec{p}_i}{(2\pi)^3 2E(\vec{p}_i)}. \quad (4)$$

Introducing the total and relative momenta of hadrons  $h^+, h^-$

$$\begin{aligned} \vec{P} &= \vec{p}_+ + \vec{p}_-, \\ \vec{p} &= \frac{\mu}{m_+} \vec{p}_+ - \frac{\mu}{m_-} \vec{p}_-, \end{aligned} \quad (5)$$

where  $\mu = m_+ m_- / (m_+ + m_-)$  is the reduced mass of  $h^+, h^-$ , the invariant matrix element can be rewritten through the new independent variables:

$$M(\vec{p}_+, \vec{p}_-; \{\vec{p}_x\}) \rightarrow M(\vec{P}, \vec{p}; \{\vec{p}_x\}). \quad (6)$$

In the rest frame of  $h^+h^-$  pair the total momentum  $\vec{P} = 0$  and the amplitude  $M(\vec{P}, \vec{p}; \{\vec{p}_x\})$  would be a monotone function of relative momentum  $p$  if the interaction in the  $h^+h^-$  pair is absent. The interaction in the final state (in particular the coulomb attraction between  $h^+$  and  $h^-$ ) violate the monotone behavior of the matrix element at small relative momenta  $p \leq \alpha\mu$  ( $\alpha = e^2/4\pi = 1/137$  is the fine structure constant) and leads to creation of coupled states of  $h^+h^-$  system ( $h^+h^-$  atoms).

To obtain the production amplitude of  $h^+h^-$  system in any certain state  $f$  one has to project the amplitude of the pair of noninteracting hadrons  $M_0(\vec{P}, \vec{p}; \{\vec{p}_x\})$  on this state

$$R_f = \int M_0(\vec{0}, \vec{p}; \{\vec{p}_x\}) \psi_f(\vec{p}) d^3p, \quad (7)$$

where  $\psi_f(\vec{p})$  is the wave function of the state  $|f\rangle$  in the momentum representation. The wave function in the coordinate representation can be obtained by Fourier transformation

$$\psi_f(\vec{p}) = \frac{1}{(2\pi)^{3/2}} \int \psi_f(\vec{r}) e^{i\vec{p}\vec{r}} d^3r \quad (8)$$

with the normalization

$$\int \psi_{f'}(\vec{r}) \psi_f(\vec{r}) d^3r = \delta_{ff'}. \quad (9)$$

The Kronecker symbol  $\delta_{ff'}$  would be interpreted as the product  $\delta_{ff'} = \delta_{nn'}\delta_{ll'}\delta_{mm'}$  when the states  $|f\rangle, |f'\rangle$  belong to the discrete spectra with the set of quantum numbers  $n, l, m$  (main, orbital, and magnetic quantum numbers, respectively) or as a product  $\delta_{ff'} = \delta(k - k')\delta_{ll'}\delta_{mm'}$  when the states  $|f\rangle, |f'\rangle$  belong to continuous spectra with the wave number  $k$  instead of main quantum number.

The analysis of the  $R_f$  dependence on quantum numbers of state  $|f\rangle$  is more transparent in coordinate representation. Introducing the relevant amplitude for the production of noninteracting hadrons in the final state

$$M_0(\vec{r}) = \frac{1}{(2\pi)^{3/2}} \int M_0(\vec{p}) e^{i\vec{p}\vec{r}} d^3p, \quad (10)$$

where  $M_0(\vec{p}) \equiv M_0(\vec{0}, \vec{p}; \{\vec{p}_x\})$ , one gets

$$R_f = \int M_0(\vec{r}) \psi_f(\vec{r}) d^3r. \quad (11)$$

Since the production of  $h^+h^-$  pairs is a result of strong interaction which decreases exponentially, the amplitude  $M_0(\vec{r})$  is nonzero only at small distances

$r \leq 1/m$ . As for the distances, where the wave function  $\psi_f(\vec{r})$  changes essentially, they are of order of Bohr radius  $r_B = 1/\mu\alpha$  for bound state and of order of  $1/k$  for continuous states in the case of pure electromagnetic interactions. Thus, we can take out the slowly varying wave function at  $\vec{r} = 0$  and put it in front of the integral (11). So we obtain

$$R_f = \psi_f(\vec{r} = 0) \int M_0(\vec{r}) d^3r = (2\pi)^{3/2} \psi_f(\vec{r} = 0) M_0(\vec{p} = 0). \quad (12)$$

This relation takes place not only for bound states but also for the creation of  $h^+h^-$  pair in the continuous state if  $kr_s \ll 1$ .

The amplitude  $R_f$  is normalized by the relation

$$(2\pi)^3 2E_f \frac{d\sigma}{d\vec{p}_f} \Big|_{\vec{p}_f=0} = \frac{1}{4\sqrt{(p_+p_-)^2 - m_+^2 m_-^2}} \int \frac{|R_f|^2 \Delta f}{2\mu(2\pi)^3} d\Phi_x, \quad (13)$$

where  $\Delta f = 1$  for discrete states and  $\Delta f = k^2 \Delta k / 2\pi^2$  in the continuous case.

Substituting the approximate relation (12) in this expression and making use of the definition (15) of the double differential cross section one gets

$$2E_f \frac{d\sigma}{d\vec{p}_f} \Big|_{\vec{p}_f=0} = \frac{(2\pi)^3 |\psi_f(\vec{r} = 0)|^2 \Delta f}{\mu} E_+ E_- \frac{d\sigma_0}{d\vec{p}_+ d\vec{p}_-} \Big|_{\vec{p}_+=\vec{p}_-=0}, \quad (14)$$

where  $\frac{d\sigma_0}{d\vec{p}_+ d\vec{p}_-} \Big|_{\vec{p}_+=\vec{p}_-=0}$  is the double differential pair production cross section without the final state interaction. As the combinations

$$E_f \frac{d\sigma}{d\vec{p}_f}, \quad E_+ E_- \frac{d\sigma_0}{d\vec{p}_+ d\vec{p}_-}$$

are relativistic invariant, from (14) it follows that in any reference frame

$$2E_f \frac{d\sigma}{d\vec{p}_f} = \frac{(2\pi)^3 |\psi_f(\vec{r} = 0)|^2 \Delta f}{\mu} E_+ E_- \frac{d\sigma_0}{d\vec{p}_+ d\vec{p}_-} \Big|_{\vec{p}_+=\alpha_+ \vec{p}_f; \vec{p}_-=\alpha_- \vec{p}_f}, \quad (15)$$

where  $\alpha_+ = m_+/(m_+ + m_-)$ ,  $\alpha_- = 1 - \alpha_+ = m_-/(m_+ + m_-)$ .

This expression connects in the closed form the inclusive production cross sections of bound and unbound states of the interacting hadron pair with the double inclusive production cross section of noninteracting pair at zero relative momentum. For the case of bound states it was obtained by L. Nemenov [2]. This expression allows one to calculate, in common approach, the relative probabilities of meson pairs production in bound and free states, which is the key point for DMA lifetime measurement [7].

### 3 Relative probability of the bound and unbound meson pairs production

The dimesoatoms lifetime measurement in experiments DIRAC at CERN based on evaluation of a DMA breakup probability in the target, which is the ratio between measured number the broken atoms in the target to the number of produced atoms. The later is extracted from the number of observed pairs in the free state using the ratio between pairs production in bound and free states [16]. Let us reconsider this ratio using equation (15). First conclusion is that the relative probability of pairs production in different states is determined solely by a two mesons wave function in the appropriate state.

Accounting that the two mesons wave function with relative orbital momenta  $L$  behaves at small relative distances  $r$  as  $\psi(r) \sim r^L$  only pairs in the S-state should be taken into account. The Coulomb wave functions of discrete  $nS$  and continuous  $kS$  states at zero relative distance reads [17]

$$\begin{aligned} |\psi_{nS}(\vec{r}=0)|^2 &= \frac{1}{\pi} \left( \frac{\mu\alpha}{n} \right)^3, \\ |\psi_{kS}(\vec{r}=0)|^2 &= C^2(k) = \frac{\pi\xi}{sh(\pi\xi)} e^{\pi\xi} = \frac{2\pi\xi}{1 - e^{-2\pi\xi}}. \end{aligned} \quad (16)$$

At large relative momentum  $k \gg \mu\alpha$  ( $\xi = \mu\alpha/k$ ) the distribution of interacting and noninteracting pairs coincides as  $C(k) \rightarrow 1$ . From the other hand at small momentum  $k \leq \mu\alpha$  ( $\xi \geq 1$ ) one can neglect the exponential term in (16) with the result:

$$|C(k)|^2 \simeq 2\pi\xi = \frac{2\pi\mu\alpha}{k}. \quad (17)$$

Thus for small relative momenta the Coulomb interaction in the final state modify significantly the distribution for the opposite charged hadrons compared to noninteracting one changing it from relative momentum independence to pole behavior:

$$\left. \frac{d^3\sigma}{d^3k} \right|_{|\vec{k}| \rightarrow 0} \rightarrow \frac{const}{|\vec{k}|}. \quad (18)$$

The production of unbound pairs with small relative momenta  $r \leq k_0 = 2\mu\alpha$  is the main background in extraction of DMA signal from experimental data. The Coulomb interaction in continuous spectra leads to the huge value of this background as compared with production of noninteracting meson pairs:

$$R_c = \frac{\int_0^{k_0} C^2(k) k^2 dk}{\int_0^{k_0} k^2 dk} \simeq \frac{6\pi\mu\alpha}{k_0} \simeq 10. \quad (19)$$

Making use the above equations one can estimate the relative probabilities of DMA production in different  $nS$  states and the relative probability of unbound pairs production in comparison with the dimesoatom production in the ground state:

$$\frac{w_{nS}}{w_{1S}} = \frac{|\psi_{nS}(\vec{r}=0)|^2}{|\psi_{1S}(\vec{r}=0)|^2} = \frac{1/\pi (\mu\alpha/n)^3}{1/\pi (\mu\alpha/1)^3} = \frac{1}{n^3}, \quad (20)$$

$$\frac{dw_{kS}}{w_{1S}} = \frac{|\psi_{kS}(\vec{r}=0)|^2}{|\psi_{1S}(\vec{r}=0)|^2} \cdot \frac{k^2 dk}{2\pi^2} = \frac{k dk}{(\mu\alpha)^2 (1 - \exp(-2\pi\mu\alpha/k))}. \quad (21)$$

The latter expression coincides with the well-known Gamov–Sommerfeld–Sakharov factor [18–20] derived for production of different pairs of oppositely charged particles. For the case of charged meson pairs production such formulas was obtained in [2].

## 4 Accuracy of the “zero radius” approximation

Thus far we exploit the fact that the DMA wave function is a slow varying function in comparison with pair production amplitude which managed by short range strong interaction. To estimate the accuracy of this approximation let us consider the amplitude of DMA production in the ground state

$$R_{1S} = \int M_0(\vec{r})\psi_{1S}(\vec{r})d^3r = 4\pi \int M_0(\vec{r})\psi_{1S}(\vec{r})r^2dr. \quad (22)$$

Making use the Coulomb wave function for ground state

$$\psi_{1S}(\vec{r}) = \frac{1}{\sqrt{\pi}} (\mu\alpha)^{3/2} \exp(-\mu\alpha r) \quad (23)$$

and choosing the Yukawa type representation [6] for the amplitude of free pairs creation<sup>1</sup>

$$M_0(\vec{r}) = \sqrt{2\pi} M_0(\vec{p}=0) \kappa^2 e^{-\kappa r} / r, \quad (24)$$

the amplitude (22) can be calculated with the result:

$$R_{1S}(\kappa) = 4\sqrt{2\pi} (\mu\alpha)^{3/2} M_0(\vec{p}=0) \frac{\kappa^2}{(\kappa + \mu\alpha)^2}. \quad (25)$$

The “zero radius” approximation corresponds to the limit  $\kappa \rightarrow \infty$ . Choosing the parameter  $\kappa$  from the interval [6]  $80 \text{ MeV} \leq \kappa \leq 140 \text{ MeV}$  one obtains

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<sup>1</sup>In the momentum space this parametrization correspond to pole like dependence:  $M_0(p) = M_0(0) \frac{\kappa^2}{\kappa^2 + p^2}$ , where  $\kappa$  is a free parameter

the estimation for the ratio of DMA production probability calculated with exact expression (25) to the approximate one calculated using “zero radius” approximation:

$$\frac{R_{1S}^2(\kappa)}{R_{1S}^2(\infty)} = \left( \frac{\kappa}{\kappa + \mu\alpha} \right)^4 = 0.950 \div 0.975. \quad (26)$$

Thus, the account of corrections on finite radius of strong interaction can reduce the DMA production cross section on  $2.5 \div 5.0\%$ .

The above consideration is the way to estimate the influence of the strong interaction on the accuracy of the cross sections in the form (15). However for the ratio (21) used for experimental data analysis, the relative effect of this influence is of order of  $10^{-3}$  only [16].

Let us generalize the above consideration to the case of any  $nS$  states. The DMA wave function for any  $nS$  state reads [17]

$$\psi_{nS}(\vec{r}) = \frac{1}{\sqrt{\pi}} \left( \frac{\mu\alpha}{n} \right)^{3/2} \exp \left( -\frac{\mu\alpha}{n} r \right) F \left( 1 - n, 2, \frac{2\mu\alpha}{n} r \right), \quad (27)$$

where  $F(1 - n, 2, 2\mu\alpha r/n)$  is the confluent hypergeometric function.

To determine the probability of DMA production to any  $nS$  state one should calculate the relevant amplitude

$$R_{nS}(\kappa) = \int M_0(\vec{r}) \psi_{nS}(\vec{r}) d^3r. \quad (28)$$

To calculate this amplitude one should compute the integral

$$\begin{aligned} I &= \int_0^\infty e^{-\lambda r} r F(1 - n, 2, \omega r) dr, \\ \lambda &= \kappa + \mu\alpha/n, \quad \omega = 2\mu\alpha/n. \end{aligned} \quad (29)$$

The general form of this integral is cited in the Appendix with the result

$$I = \frac{1}{\lambda^2} \left( 1 - \frac{\omega}{\lambda} \right)^{n-1}. \quad (30)$$

The amplitude (28) for any  $nS$  state production can be presented in the form

$$R_{nS}(\kappa) = 4\sqrt{2}\pi \left( \frac{\mu\alpha}{n} \right)^{3/2} M_0(\vec{p}=0) \frac{\kappa^2}{(\kappa + \mu\alpha/n)^2} \left( \frac{\kappa n - \mu\alpha}{\kappa n + \mu\alpha} \right)^{n-1}. \quad (31)$$

This expression allows one to estimate the corrections to the “zero radius” approximation for any  $nS$  state production in proton-proton collisions.

## 5 Amplitude of DMA transition from 1S to 2P state

Let us consider the inelastic transition from the DMA ground state 1S to the first bound state with nonzero orbital, i.e. 2P state. The selection rules allow only transitions to the states with  $|m| = 1$  (transition to the state with  $m = 0$  is forbidden). The amplitude for such transition reads

$$\begin{aligned} A_{fi}(\vec{q}) &= \int d^2s f(\vec{q}, \vec{s}) h_{fi}(\vec{s}), \\ f(\vec{q}, \vec{s}) &= \frac{i}{2\pi} \int d^2b \left[ 1 - e^{i\Delta\chi(\vec{b}, \vec{s})} \right] e^{i\vec{q}\vec{b}}, \\ h_{fi}(\vec{s}) &= \int_{-\infty}^{\infty} dz \psi_f^*(\vec{r}) \psi_i(\vec{r}), \quad \vec{r} = (\vec{s}, z) \end{aligned} \quad (32)$$

with the wave functions  $\psi_{i(f)}(\vec{r})$  of the initial ( $|i\rangle = |1S\rangle$ ) and final  $|f\rangle = |2P^{(\pm 1)}\rangle$  states:

$$\begin{aligned} \psi_i(\vec{r}) &= \frac{(\mu\alpha)^{3/2}}{\sqrt{\pi}} e^{-\mu\alpha r}, \\ \psi_f(\vec{r}) &= \frac{(\mu\alpha)^{5/2} \vec{\epsilon}_+ \vec{r}}{4\sqrt{2\pi}} e^{-\mu\alpha r/2}, \quad \vec{\epsilon}_{\pm} = \vec{e}_x \pm i\vec{e}_y. \end{aligned} \quad (33)$$

Substituting these expressions in (32) we obtain

$$h_{fi}(\vec{s}) = \frac{(\mu\alpha)^4}{2\sqrt{2\pi}} (\vec{\epsilon}_+ \vec{s}) s K_1(\tilde{\mu}s), \quad \tilde{\mu} = \frac{3\mu\alpha}{2}. \quad (34)$$

From the other hand due to wave functions orthogonality we have

$$\int h_{fi}(\vec{s}) d^2s = 0. \quad (35)$$

At small transfer momenta  $q^2 \ll \tilde{\mu}^2$  only single photon exchange is essential. In this case the integration in (32) can be done with the result

$$\begin{aligned} A_{fi}(\vec{q}) &= A_{fi}^{1B}(\vec{q}) \left\{ 1 + O \left[ \frac{q^2}{\tilde{\mu}^2} \ln \left( \frac{\tilde{\mu}^2}{q^2} \right) (Z\alpha)^2 \right] \right\}, \\ A_{fi}^{1B}(\vec{q}) &= i\alpha Z \sqrt{2} (\mu\alpha)^4 \frac{(\vec{\epsilon}_{\mp} \vec{q})}{q^2} \cdot \tilde{\mu} \left( \tilde{\mu}^{-1} \frac{\partial}{\partial \tilde{\mu}} \right)^2 \frac{1}{\tilde{\mu}^2 + q^2/4}. \end{aligned} \quad (36)$$

In this expression the factor  $(\vec{\epsilon}_{\mp} \vec{q})/q^2$  appears as we disregard the screening effect considering the transition ( $1S \rightarrow 2P^{\pm 1}$ ) at transfer momentum  $q$  much larger than the inverse Born radius of the target atoms  $\lambda_B = 1/R_{sc} \approx m_e \alpha Z^{1/3}$ .



The Born amplitude generalized to take into account the screening effect takes the form

$$\begin{aligned} A_{fi}^{1B}(\vec{q}) &= i\alpha Z\sqrt{2}(\mu\alpha)^4 \frac{(\vec{\epsilon}_{\mp}\vec{q})}{q^2 + \lambda_B^2} \mu \left( \tilde{\mu}^{-1} \frac{\partial}{\partial \tilde{\mu}} \right)^2 \frac{1}{\tilde{\mu}^2 + q^2/4} \\ &= i\frac{3\sqrt{2}}{4}(4\mu\alpha)^5 Z \frac{(\vec{\epsilon}_{\mp}\vec{q})}{(q^2 + \lambda_B^2)((3\mu\alpha)^2 + q^2)^3}. \end{aligned} \quad (37)$$

In this way the differential cross section of the dimesoatom transition from 1S to 2P state reads

$$d\sigma = \frac{9}{8}Z^2(4\mu\alpha)^{10} \frac{q^2 d^2q}{(q^2 + (m_e\alpha Z^{1/3})^2)^2 (q^2 + (3\mu\alpha)^2)^6}. \quad (38)$$

## 6 Summary

In this paper we consider the production of oppositely charged meson pairs in collision of relativistic particles and obtain the general expressions for amplitudes and cross sections for the creation of any bound states (dimesoatoms) and continuous states accounting for the electromagnetic interaction in the final state. We derive the general expression for such type production using the coordinate representation for relevant wave functions and amplitudes. We obtain the expressions which allow one to estimate the relative probability of bound and continuous states production for any mesons pair. The amplitude for any bound state production beyond the widely used in the literature “zero radius” approximation has been obtained. Finally we obtain the analytical expression for the cross section of dimesoatom transition from 1S to 2P state.

## Appendix

To calculate the integral

$$I = \int_0^\infty e^{-\lambda r} r^{b-1} F(a, b; \omega r) dr \quad (39)$$

it is convenient to use the representation of confluent hypergeometric function in the form of contour integral

$$F(a, b; z) = -\frac{1}{2\pi i} \frac{\Gamma(1-a)\Gamma(b)}{\Gamma(b-a)} \oint_C e^{tz} (-t)^{a-1} (1-t)^{b-a-1} dt. \quad (40)$$

The contour  $C$  begin at  $t = 1$ , bypass the point  $t = 1$  counterclockwise and return at point  $t = 1$ .

This representation allows one to carry on the integration with the result

$$\begin{aligned} I &= -\frac{1}{2\pi i} \frac{\Gamma(1-a)\Gamma(b)^2}{\Gamma(b-a)\lambda^b} \oint_C \frac{(-t)^{a-1}(1-t)^{b-a-1}}{(1-\omega t/\lambda)^b} dt \\ &= \frac{\Gamma(b)^2}{\lambda^b} {}_2F_1\left(a, b; b; \frac{\omega}{\lambda}\right), \end{aligned} \quad (41)$$

where the hypergeometric Gauss function  ${}_2F_1(a, b; c; z)$  is determined by the series

$$\begin{aligned} {}_2F_1(a, b; c; z) &\equiv F(a, b; c; z) = \sum_{j=0}^{\infty} \frac{(a)_j \cdot (b)_j}{(c)_j \cdot j!} z^j, \\ (a)_j &= \frac{\Gamma(a+j)}{\Gamma(a)} = a(a+1) \dots (a+j-1), \\ (b)_j &= \frac{\Gamma(b+j)}{\Gamma(b)} = b(b+1) \dots (b+j-1), \\ (c)_k &= \frac{\Gamma(c+k)}{\Gamma(c)} = c(c+1) \dots (c+k-1), \\ (a)_0 &= (b)_0 = (c)_0 = 1, \end{aligned} \quad (42)$$

or with the representation through the contour integral

$$F(a, b; c; z) = -\frac{1}{2\pi i} \frac{\Gamma(1-a)\Gamma(c)}{\Gamma(c-a)} \oint_C (-t)^{a-1} (1-t)^{c-a-1} (1-tz)^{-b} dt. \quad (43)$$

The contour  $C$  in (43) is the same as in the definition of the confluent hypergeometric function. Substitution  $t \rightarrow t/(1-z+zt)$  in the integral (43) leads to the relation between the hypergeometric functions at different values of variables  $z$  and  $z/(z-1)$ :

$$F(a, b; c; z) = (1-z)^{-a} F\left(a, c-b; c; \frac{z}{z-1}\right).$$

Applying this relation to the hypergeometric function one obtains

$$F\left(a, c; c; \frac{\omega}{\lambda}\right) = \left(1 - \frac{\omega}{\lambda}\right)^{-a} F\left(a, 0; c; \frac{\omega}{\omega - \lambda}\right),$$

where  $F(a, 0; c; z) = 1$  as  $(b)_j = 0$  at  $j \neq 0$  and  $b = 0$  in definition of hypergeometric function by series.

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